Kruskal–Wallis H test

By Dr. Satyabrata Dash
Principal,
SMIT-PGSMS

Kruskal-Wallis Test or H-Test is the non-parametric and a valuable alternative to a one-way ANOVA test (F-test) when the normality and equality of variance assumptions are violated.

The usual ANOVA test is used to test the equality of several population means and is based on the fundamental assumption that the population from which samples are drawn are normally distributed.

Assumptions:

- There are at least three independently drawn random samples.
- Each sample has at least 5 observations, $n_i \geq 5$.

Kruskal-Wallis Test is used to test the Null Hypothesis if k independent samples, $k \geq 3$, have been drawn from the population which have identical distribution, and does not require the condition of normality of the populations.

Null Hypothesis H_0:

the $k-$ independent samples have been drawn from populations which are identically distributed.

Alternative Hypothesis H_1:

the $k-$ independent samples have been drawn from populations which are NOT identically distributed.

Alternatively:

Null Hypothesis H_0:

$$\mu_1 = \mu_2 = \cdots = \mu_k$$

Alternative Hypothesis H_1:

At least one of the μ_i's is different from others.

Notation:

k: number of samples

n_i: Size of the ith sample; $i = 1,2,\dots,k$

$n = n_1 + n_2 + \cdots + n_k$

Total number of observation in k samples

T_i: sum of the ranks of the ith sample;

Sample	Observations							
1	…	…	…	…	…			
2	…	…	…	…	…	…	…	
…	…	…	…	…	…	…	…	…
k	…	…	…	…	…	…	…	

Rank from smallest (1) to the largest (n).

If there is a tie, then take an average.

Procedure:

Step 1: Define Null Hypothesis (H_0) and Alternative Hypothesis (H_1)

Step 2: Rank the sample observations in the combined series and

Compute T_i: sum of the ranks of the ith sample

Step 3: Compute Test statistics, H-value:

Under H_0,

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} \frac{T_i^2}{n_i} - 3(n+1)$$

Step 4: Conclusion:

Take the tabulated value of $\chi^2_{(k-1)}(\alpha)$.

If the calculated value of $H > \chi^2_{(k-1)}(\alpha)$, we REJECT H_0, otherwise, we fail to reject H_0.

Example: A company's trainees are randomly divided into three groups of 10 each and are given a course in management skill by three different methods. At the end of the training period, they are given a test and their scores are as follows:

Method A	99	64	101	85	79	88	97	95	90	100
Method B	83	102	125	61	91	96	94	89	93	75
Method C	89	98	56	105	87	90	87	101	76	89

Use Kruskal-Wallis test to determine at 5% level of significance if the three methods are equally effective.

Solution: Step 1:

Null Hypothesis: $H_0: \mu_A = \mu_B = \mu_C$,

i.e., three methods are equally effective.

Alternative Hypothesis: H_1: At least two of μ's are different,

i.e., the three methods are not equally effective.

Step 2: Compute the ranks

Sample:	C	B	A	B	C	A	B	A	C	C	A	B	C	C
Observation:	56	61	64	75	76	79	83	85	87	87	88	89	89	89
Rank:	1	2	3	4	5	6	7	8	9.5	9.5	11	13	13	13

Sample:	A	C	B	B	B	A	B	A	C	A	A	A	C	B	C	B
Observation:	90	90	91	93	94	95	96	97	98	99	100	10	101	102	105	125
Rank:	15.5	15.5	17	18	19	20	21	22	23	24	25	26.5	26.5	28	29	30

OR

Method A	99 (24)	64 (3)	101 (26.5)	85 (8)	79 (6)	88 (11)	97 (22)	95 (20)	90 (15.5)	100 (25)
Method B	83 (7)	102 (28)	125 (30)	61 (2)	91 (17)	96 (21)	94 (19)	89 (13)	93 (18)	75 (4)
Method C	89 (13)	98 (23)	56 (1)	105 (29)	87 (9.5)	90 (15.5)	87 (9.5)	101 (26.5)	76 (5)	89 (13)

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$n_1 = 10;\ \ n_2 = 10\ ;\ n_3 = 10;\quad n = n_1 + n_2 + n_3 = 30$

$T_1 = 24 + 3 + 26.5 + 8 + 6 + 11 + 22 + 20 + 15.5 + 25 = 161$

$T_2 = 7 + 28 + 30 + 2 + 17 + 21 + 19 + 13 + 18 + 4 = 159$

$T_3 = 13 + 23 + 1 + 29 + 9.5 + 15.5 + 9.5 + 26.5 + 5 + 13 = 145$

Step 3: Compute Test Statistics H

Under H_0, the Kruskal-Wallis test is

$$H = \frac{12}{n(n+1)}\sum_{i=1}^{3}\frac{T_i^2}{n_i} - 3(n+1)$$

$$= \frac{12}{30(30+1)}\left[\frac{(161)^2}{10} + \frac{(159)^2}{10} + \frac{(145)^2}{10}\right] - 3(30+1)$$

$$= 0.196$$

Chi-square Distribution Table

d.f.	.995	.99	.975	.95	.9	.1	.05	.025	.01
1	0.00	0.00	0.00	0.00	0.02	2.71	3.84	5.02	6.63
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22

Step 4: Conclusion:

The degree of freedom $= k - 1 = 3 - 1 = 2$

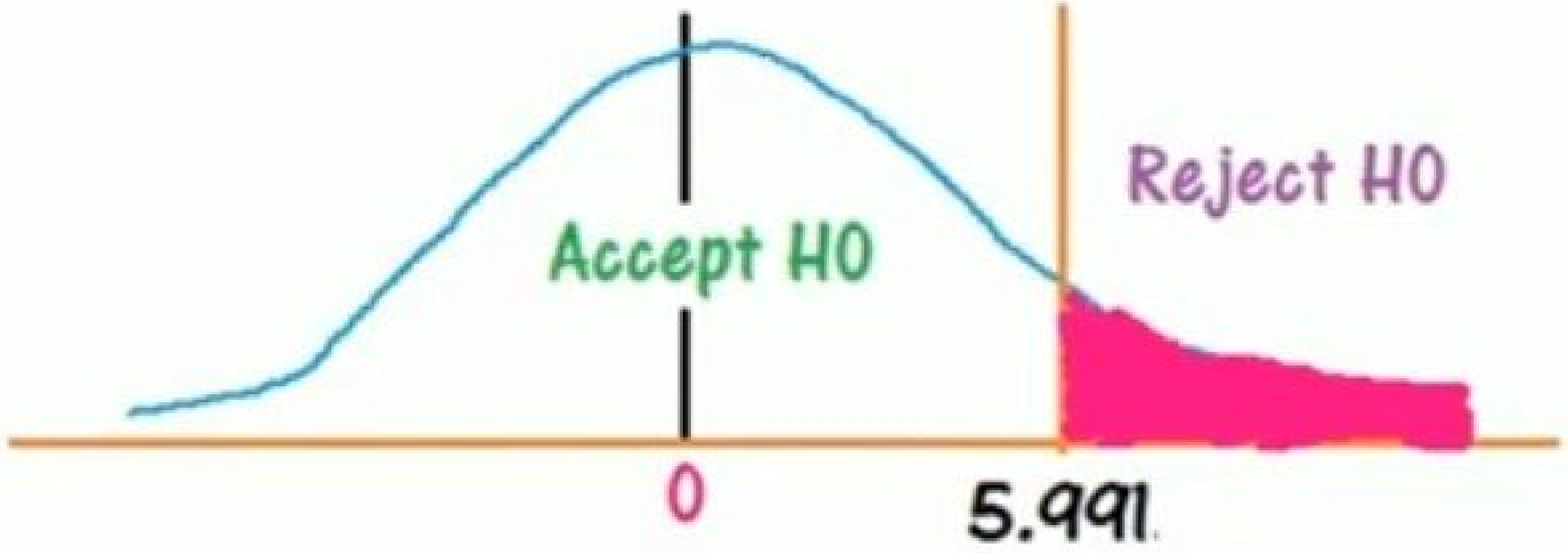
The tabulated (critical) value of $\chi^2(0.05)$ for 2 d.f. is 5.991

Since the calculated H-value

$0.196 < 5.991$,

so it is not significant and

we fail to reject H_0.



Hence, we conclude that all the three methods are equally effective, i..e, they do not differ significantly.

Example: The following data were collected which are the achievement test scores for four different groups of students, each group taught by a different teaching technique. The objective of the experiment is to test the hypothesis of no difference in the population distributions of achievement test scores versus the alternative that they differ in locations; that is; at least one of the distribution is shifted above the others. Conduct the test using the Kruskal-Wallis H-test with $\alpha = 0.05$.

I	II	III	IV
65	75	59	94
87	69	78	89
73	83	67	80
79	81	62	88
81	72	83	96
69	79	76	
	90		

Solution:

Step 1:

Null Hypothesis:

$$H_0: \mu_I = \mu_{II} = \mu_{III} = \mu_{IV},$$

i.e., four teaching techniques are equally effective.

Alternative Hypothesis: H_1:

At least two of μ's are different, i.e., the four teaching techniques are not equally effective.

Step 2: Rank the numbers

I	II	III	IV
65 (3)	75 (9)	59 (1)	94 (23)
87 (19)	69 (5.5)	78 (11)	89 (21)
73 (8)	83 (17.5)	67 (4)	80 (14)
79 (12.5)	81 (15.5)	62 (2)	88 (20)
81 (15.5)	72 (7)	83 (17.5)	96 (24)
69 (5.5)	79 (12.5)	76 (10)	
	90 (22)		

$n_1 = 6; n_2 = 7;$

$n_3 = 6; \; n_4 = 5$

$\therefore n = n_1 + n_2 + n_3 + n_4$

$= 24$

Sum of the Ranks

$T_1 = 63.5$

$T_2 = 89$

$T_3 = 45.5$

$T_4 = 102$

Chi-square Distribution Table

d.f.	.995	.99	.975	.95	.9	.1	.05	.025	.01
1	0.00	0.00	0.00	0.00	0.02	2.71	3.84	5.02	6.63
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09
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12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22

Step 3: Compute Test Statistics H

Under H_0, the Kruskal-Wallis test is

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{3} \frac{T_i^2}{n_i} - 3(n+1)$$

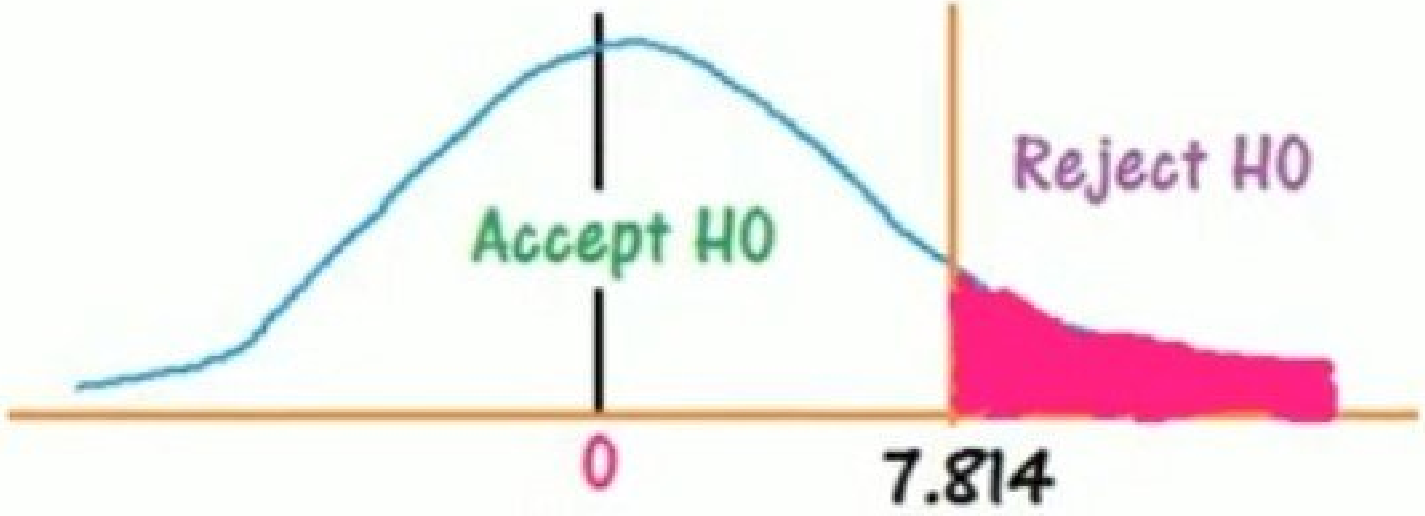
$$= \frac{12}{24(24+1)}\left[\frac{(63.5)^2}{6} + \frac{(89)^2}{7} + \frac{(45.5)^2}{6} + \frac{(102)^2}{5}\right] - 3(25)$$

$$= 9.5891$$

Step 4: Conclusion:

The degree of freedom $= k - 1 = 4 - 1 = 3$

Critical value of $\chi^2(0.05)$ for 3 d.f. is 7.814



Since $9.5891 > 7.814$,

So it is highly significant and we reject H_0.

Therefore, we conclude that there is a significant difference in the distributions of achievement test scores for the four teaching techniques.